Transient quantum fluctuation theorems

Peter Talkner
Institut für Physik, Universität Augsburg

Peking University, September 2015

Acknowledgments

Michele Campisi

Prasanna Venkatesh

Peter Hänggi

Gentaro Watanabe

Manuel Morillo

Juyeon Yi

Overview

- Transient fluctuation theorems
- Work
- Quantum work statistics and fluctuation relations
- Generalized measurements and Crooks relation
- Gaussian energy measurements and modified fluctuation relations
- A work meter
- Summary

Colloquium: Quantum fluctuation relations: Foundations and applications

Michele Campisi, Peter Hänggi, and Peter Talkner
Rev. Mod. Phys. **83**, 771 – Published 6 July 2011; Erratum Rev. Mod. Phys. **83**, 1653 (2011)

PERSPECTIVE | INSIGHT

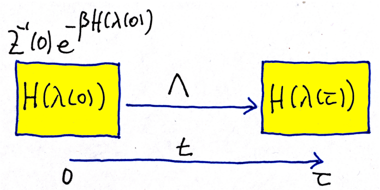
The other QFT

P. Hänggi, P. Talkner,

Nature Physics **11**, 108 (2015).

Jarzynski equality

$H(\lambda)$: system Hamiltonian, λ: classical parameter



$\Lambda = \{\lambda(t) | 0 \leq t \leq \tau\}$: protocol

w: Work performed on the system

$$\boxed{\langle e^{-\beta w} \rangle = e^{-\beta \Delta F}}$$

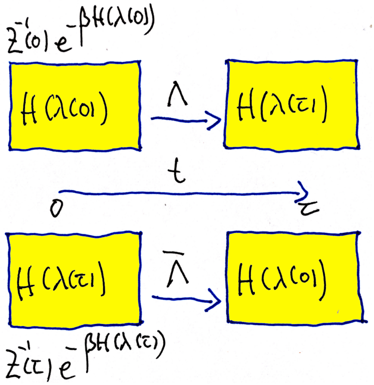
Jarzynski, PRL **78**, 2690 (1997).

$\langle \cdot \rangle$: average over realizations of the same protocol

$\Delta F = F(\tau) - F(0)$, $F(t) = -\beta^{-1} \ln Z(t)$, $Z(t) = \mathrm{Tr} e^{-\beta H(\lambda(t))}$

Jensen's inequality $\Longrightarrow$ $\langle w \rangle \geq \Delta F$ 2nd law

Crooks relation



$\Lambda = \{\lambda(t)|0 \leq t \leq \tau\}$ forward protocol

$\bar{\Lambda} = \{\lambda(\tau - t)|0 \leq t \leq \tau\}$ backward protocol

$$p_\Lambda(w) = e^{-\beta(\Delta F - w)} p_{\bar{\Lambda}}(-w)$$

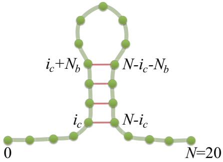
G.E. Crooks, PRE **60**, 2721 (1999)

$p_\Sigma(w)$: pdf of work w during protocol $\Sigma = \Lambda, \bar{\Lambda}$

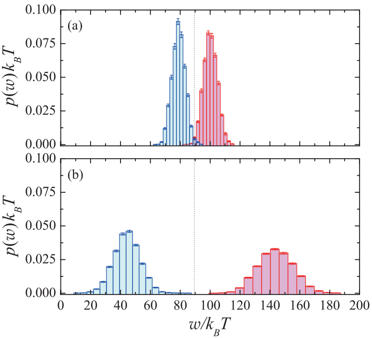
Crooks $\Rightarrow$ Jarzynski

Applications

Pulling macromolecules in order to determine free energy differencies between different confirmations: Liphardt et al., Science **296**, 1832 (2002); Collin et al., Nature **437**, 231 (2005); Douarche et al., Europhys. Lett. **70**, 593 (2005).



S. Kim, Y.W. Kim, P. Talkner, J.Yi, Phys. Rev. E **86**, 041130 (2012).



Jarzynski: $\Delta F = -\beta^{-1} ln\langle e^{-\beta w}\rangle$

Crooks: $p_\Lambda(w) = e^{-\beta(\Delta F - w)} p_{\bar{\Lambda}}(-w) \;\Rightarrow p_\Lambda(w)$ and $p_{\bar{\Lambda}}(-w)$ cross at $w = \Delta F$

Work

Classical thermally isolated system:

$$\begin{aligned} w &= H(z(\tau), \lambda(\tau)) - H(z, \lambda(0)) \\ &= \int_0^\tau dt \frac{dH(z(t), \lambda(t))}{dt} \\ &= \int_0^\tau dt \frac{\partial H(z(t), \lambda(t))}{\partial \lambda} \dot{\lambda}(t) \end{aligned}$$

Note that a proper gauge must be used in order that the Hamiltonian yields the energy.

$z(t)$: solution of the Hamiltonian equations of motion

$$\dot{z}(t) = \{H(z(t), \lambda(t)), z(t)\}$$

with $z(0) = z$: point in phase space

Work characterizes a process; it comprises information from states at distinct times. Hence it is not an observable. As such it would only present information about the state at a single instant of time.

The measurement of the quantum versions of power- and energy-based work definitions requires different strategies.

1. Two energy measurements:

One at the beginning, the other at the end of the protocol yield eigenvalues $e_n(0)$ and $e_m(\tau)$ of $H(\lambda(0))$ and $H(\lambda(\tau))$.

$$w^e = e_m(\tau) - e_n(0) \implies \text{fluctuation theorems.}$$

$$H(\lambda(t)) = \sum_n e_n(t)\Pi_n(t)$$

$\Pi_n(t)$: Projection operator on the eigenstate of $H(\lambda(t))$ with eigenenergy $e_n(t)$.

J. Kurchan, arXiv:cond-mat/0007360 (2000);
H. Tasaki, arXiv:cond-mat/0009244 (2000);
P. Talkner, E. Lutz, P. Hänggi, Phys. Rev E **75**, 050102 (2007).

2. Power-based work:

Requires a continuous measurement of power.
E.g. for $H(\lambda) = H_0 + \lambda Q$, a continuous observation of the generalized coordinate Q is required leading to a freezing of the systems dynamics in an eigenstate of Q.

$$w_N^p = \sum_{k=1}^{N} \dot{\lambda}(t_k) q_{\alpha_k} \frac{\tau}{N-1}, \quad Q = \sum_\alpha q_\alpha \Pi_\alpha^Q$$

Fluctuation theorems hold only if $[H_0, Q] = 0$ or equivalently $[H(\lambda(t)), H(\lambda(s))] = 0$ for all $t, s \in (0, \tau)$.
Hence the equivalence of the power- and energy-based work definitions for classical systems fails to hold in quantum mechanics.

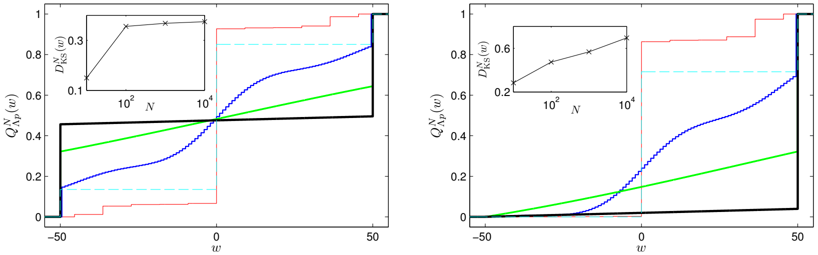
Example: Landau-Zener

$$H(t) = \frac{vt}{2}\sigma_z + \Delta\sigma_x, \quad -\tau/2 \le t \le \tau/2$$

possible work-values:

$$\mathcal{W}^e = \{-E_0, 0, E_0\}\,, \; E_0 = \left((v\tau/2)^2 + \Delta^2)\right)^{1/2} \qquad \text{energy-based}$$

$$\mathcal{W}^e = \left\{ \frac{v\tau}{2(N+1)} g, g = -N, -N+2, \ldots, N \right\} \qquad \text{power-based}$$



$$v = 5\Delta^2/\hbar, \quad \tau = 20\hbar/\Delta$$

$N = 10,\ 10^2,\ 10^3,\ 10^4,$ energy based.

B.P. Venkatesh, G. Watanabe, P. Talkner, New J. Phys. **17**, 075018 (2015).

Work pdf from two energy measurements

$$p_\Lambda(w) = \sum_{n,m} \delta\big(w - e_m(\tau) + e_n(0)\big)\, p_\Lambda(m,n) \ : \qquad \text{work pdf}$$

$$H(\lambda(t)) = \sum_n e_n(t)\Pi_n(t) \qquad \text{spectral rep.}$$

$$p_\Lambda(m,n) = \mathrm{Tr}\,\Pi_m(\tau)U_\Lambda \Pi_n(0)\rho(0)\Pi_n(0)U_\Lambda^\dagger \qquad \text{joint prob.}$$

$$\Lambda = \{\lambda(t) | 0 \leq t \leq \tau\} \ : \qquad \text{protocol}$$

$$U_\Lambda = U_{\tau,0}(\Lambda)\,, \quad i\hbar \frac{\partial}{\partial t} U_{t,s}(\Lambda) = H(\lambda(t))U_{t,s}(\Lambda)\,, \quad U_{s,s}(\Lambda) = \mathbb{1}$$

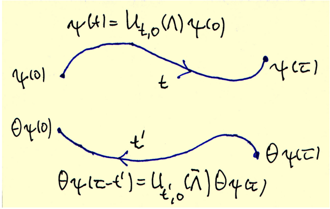
CROOKS RELATION, $p_\Lambda(w) = e^{-\beta(\Delta F - w)} p_{\bar{\Lambda}}(-w)$, follows from two requirements

(i) The diagonal elements of the initial states of the forward and the backward process are both given by Boltzmann factors for the respective parameters $\lambda(0)$ and $\lambda(\tau)$ and same temperature β^{-1}, i.e. the diagonal elements of the respective density matrix $\rho(t)$ is given by

$$\Pi_n(t)\rho(t)\Pi_n(t) = Z^{-1}(t) e^{-\beta e_n(t)} \Pi_n(t)$$
$$Z(t) = \sum_n e^{-\beta e_n(t)} d_n(t) \quad t = 0, \tau$$
$$d_n(t) = \mathrm{Tr}\Pi_n(t) \quad \text{degeneracy of } e_n(t)$$

(ii) Time-reversal invariance

time-reversal invariance



$$H(\lambda(t)) = \theta H(\epsilon_\lambda \lambda(t))\theta^\dagger$$
$$\Longrightarrow$$
$$U_{s,t}(\Lambda) = U^{-1}_{t,s}(\Lambda) = \theta^\dagger U_{\tau-s,\tau-t}(\bar{\Lambda})\theta$$

D. Andrieux, P. Gaspard, Phys. Rev. Lett. **100**, 230404 (2008).

$$P_\Lambda(m|n)d_n(0) = P_{\bar{\Lambda}}(n|m)d_m(\tau)\ ,$$ generalized detailed balance

$$P_\Lambda(m|n) = \mathrm{Tr}\Pi_m(\tau)U_\Lambda \Pi_n(0)U^\dagger_\Lambda/dn(0)\ ,$$ transition probability

$$d_n(t) = \mathrm{Tr}\Pi_n(t)\ ,\quad t = 0, \tau\ :\ \#\text{ of states with } e_n(t)$$

gdb + canonical initial states ⇔ Crooks relation

P. Talkner, M. Morillo, J. Yi, P. Hänggi, New J. Phys. 15, 095001 (2013).

Experiments

The classical fluctuation relations are experimentally confirmed for mechanical, electrical and molecular systems and are the basis of a method to determine free energy differences.

In quantum systems, projective energy measurements pose a severe problem.

Proposal of an experiment:

G. Huber, F. Schmidt-Kaler, S. Deffner, E. Lutz, Phys. Rev. E **101**, 070403 (2008).

First experiment:

S. An et al. Nat. Phys. **11**, 193 (2015).

Alternative method avoiding projective measurements:

R. Dorner, S.R. Clark, L. Heaney, R. Fazio, J. Goold, V. Vedral, Phys. Rev. Lett. **110**, 230601 (2013); L. Mazzola, G. De Chiara, M. Paternostro, Phys. Rev. Lett. **110**, 230602 (2013); M. Campisi, R. Blattmann, S. Kohler, D. Zueco, P. Hänggi, New J. Phys. **15**, 105028 (2013).

Experimental confirmation:

T. Batalhão et al., Phys. Rev. Lett. **113**, 140601 (2014).

Can generalized measurements be of help?

Generalized measurements

In QM the measurement of an observable $\mathcal{A}$ in a state ρ
(i) assigns to $\mathcal{A}$ a real value a with probability p_a
(ii) transforms the state ρ of the system at the instant before the measurement to a new state after the measurement:

$$\rho_a^{pm} = \phi_a(\rho)/p_a$$

The *measurement operation* $\phi_a : TC(\mathcal{H}) \to TC(\mathcal{H})$ is a linear, positive and contractive map.
Hence it can be represented as (K. Kraus, *States, Effects and Operators*, Springer 1983)

$$\phi_a(\rho) = \sum_\alpha M_\alpha^a \rho M_\alpha^{a\dagger}$$

$M_\alpha^a \in B(\mathcal{H})$: Kraus operators

$$p_a = \mathrm{Tr}\phi_a(\rho) : \text{ probability to find } a$$

Energy measurements

$$H(\lambda(t)) = \sum_n e_n(t)\Pi_n(t)$$

ϕ_n^t measurement operation of the observable $\Pi_n(t)$.
For a *projective measurement* ϕ_n^t becomes

$$\phi_n^t(\rho) = \Pi_n(t)\rho\Pi_n(t) \quad \text{Lüders-von-Neumann rule}$$

It then is a REPRODUCIBLE and ERROR-FREE measurement:

$$\phi_n^t(\phi_n^t(\rho)) = \phi_n^t(\rho) \qquad \text{reproducible}$$
$$p(n|m) \equiv \mathrm{Tr}\phi_n^t(\Pi_m(t)/d_m(t)) = \delta_{n,m} \quad \text{error-free}$$

If $\Pi_n(t) = |n;t\rangle\langle n;t|$ a single Kraus operator suffices to represent any error-free measurement operation, i.e. $\phi_n(\rho) = M_n\rho M_n^\dagger$. The Kraus operator then is of the form

$$M_n = |\psi_n(t)\rangle\langle n;t|$$

Staring from a canonical initial state and using generalized energy measurements one obtains for the work pdf

$$p_\Lambda^\phi(w) = \sum_{m,n} \delta(w - e_m(\tau) + e_n(0)) p_\Lambda^\phi(m,n)$$

$$\begin{aligned} p_\Lambda^\phi(m,n) &= \mathrm{Tr}\phi_m^\tau\left(U_\Lambda \phi_n^0 \left(\rho(0) \right) U_\Lambda^\dagger \right) \\ &= Z^{-1}(0) \sum_{n'} e^{-\beta e_{n'}(0)} \mathrm{Tr}\phi_m^\tau\left(U_\Lambda \phi_n^0 \left(\Pi_{n'}(0) \right) U_\Lambda^\dagger \right) \end{aligned}$$

For a class of "typical" protocols with protocol and temperature independent energy measurements the validity of the Crooks relation

$$p_\Lambda^\phi(w) = e^{-\beta(\Delta F - w)} p_{\bar\Lambda}^{\bar\phi}(-w)$$

excludes all energy measurement operations but projective ones. For the proof of a similar statement see

B.P. Venkatesh, G. Watanabe, P. Talkner, New J. Phys. **16**, 015032 (2014).

Continuous energy measurements

Gaussian energy measurement

$$M_E(t) = \frac{1}{(2\pi\mu^2)^{1/4}} \exp\left(\frac{1}{4\mu^2}(H(\lambda(t)) - E)^2\right)$$
$$M_E^\dagger(t) = M_E(t)$$

Gaussian failure distribution:

$$\begin{aligned} p_t(E|n) &= \mathrm{Tr} M_E^2(t)\Pi_n(t)/d_n(t) \\ &= \frac{1}{\sqrt{2\pi\mu^2}} \exp\left(\frac{1}{2\mu^2}(e_n(t) - E)^2\right) \end{aligned}$$

with variance μ^2 which is independent of n.
Joint pdf to find the energy E and E' in the beginning and at the end of the protocol Λ, respectively, for a canonical ρ_0

$$\begin{aligned} p_\Lambda^{\mathrm{Gauss}}(E', E) &= \mathrm{Tr} M_{E'}^2(\tau) U(\Lambda) M_E(0)\rho(0) M_E(0) U^\dagger(\Lambda) \\ &= \sum_{m,n} p_\tau(E'|m) p_0(E|n) p_\Lambda(m, n) \end{aligned}$$

Work pdf:

$$p_\Lambda^{\text{Gauss}}(w) = \int dE' dE \delta(w - E' + E) P_\Lambda^{\text{Gauss}}(E', E)$$
$$= \int \frac{dw'}{\sqrt{4\pi\mu^2}} e^{-(w-w')^2/(4\mu^2)} p_\Lambda(w)$$

Generalized Crooks relation

$$p_\Lambda^{\text{Gauss}}(w - \beta\mu^2) = e^{-\beta(\Delta F - w)} p_{\bar{\Lambda}}^{\text{Gauss}}(-w - \beta\mu^2)$$

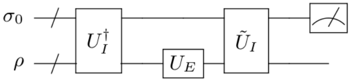
Generalized Jarzynski equality:

$$\langle e^{-\beta w} \rangle = e^{-\beta(\Delta F - \mu^2 \beta)}$$

G. Watanabe, B.P. Venkatesh, P. Talkner, Phys. Rev. E **89**, 052116 (2014).

Work meter

Device proposed by De Chiara, Roncaglia and Paz, (New J. Phys. **17**, 035004 (2015)) gives work by a single measurement.



$$U_I = e^{i\kappa H(\lambda(0))P/\hbar}$$

$$\tilde{U}_I = e^{i\kappa H(\lambda(\tau))P/\hbar}$$

$$U_E \equiv U_\Lambda \,, \;\; \rho = \rho(o)$$

P: momentum conjugate to the pointer-position $X = \int dx \, x \, \mathbb{Q}_x$, $\mathbb{Q}_x = |x\rangle\langle x|$.

$$\begin{aligned} p_\Lambda^X(x) &= \mathrm{Tr}_{S+P} \mathbb{Q}_x \tilde{U}_I U_\Lambda U_I^\dagger \rho(0) \otimes \sigma_0 U_I U_\Lambda^\dagger \tilde{U}_I \\ &= \sum_{\substack{m \\ n,n'}} p_\Lambda(m,n,n') \sigma_0(x - \kappa w_{m,n}, x - \kappa w_{m,n'}) \end{aligned}$$

$$p_\Lambda(m,n,n') = \mathrm{Tr}_S \Pi_m(\tau) U_\Lambda \Pi_n(0) \rho(0) \Pi_{n'}(0) U^\dagger$$

$$\sigma_0(x,y) = \langle x|\sigma_0|y\rangle \,, \quad w_{m,n} = e_m(\tau) - e_n(0)$$

Gaussian pointer

Gaussian pure state as initial state of the pointer:

$$\psi_0(x) = \frac{1}{(2\pi\mu^2)^{1/4}} e^{-x^2/(4\mu^2)} \Rightarrow \sigma_0(x,y) = \frac{1}{\sqrt{2\pi\mu^2}} e^{-(x^2+y^2)/(4\mu^2)}$$

$\langle x \rangle_0 = 0$, hence $w = x/\kappa$ is an unbiased work estimate

$$p_\Lambda^{\text{pointer}}(w) = \sum_{\substack{m\\ n,n'}} \frac{1}{\sqrt{2\pi\mu_w^2}} e^{-\frac{1}{8\mu_w^2}[e_n(0)-e_{n'}(0)]^2}$$

$$\times\, e^{-\frac{1}{2\mu_w^2}\left[w - e_m(\tau) + \frac{1}{2}(e_n(0)+e_{n'}(0))\right]^2} p_\Lambda(m,n,n')$$

$$\mu_w = \frac{\mu}{\kappa}$$

Special cases

(1) Stationary initial state: $[H(\lambda(0)), \rho(0)] = 0$ gives

$$\begin{aligned} p_\Lambda(m, n, n') &= \mathrm{Tr}\Pi_m(\tau) U_\Lambda \Pi_n(0)\rho(0)\Pi_{n'}(0) \\ &= p_\Lambda(m, n)\delta_{n,n'} \end{aligned}$$

$$\begin{aligned} p_\Lambda^{\text{pointer}}(w) &= \sum_{m,n} \frac{1}{\sqrt{2\pi\mu_w^2}} e^{-\frac{1}{2\mu_w^2}[w - e_m(\tau) + e_n(0)]^2} \\ &= \int dw' \frac{1}{\sqrt{2\pi\mu_w^2}} e^{-\frac{1}{2\mu_w^2}(w - w')^2} p_\Lambda(w') \end{aligned}$$

(2) accurate measurement

$$p_\Lambda^{\text{pointer}}(w) = \sum_{\substack{m \\ n,n'}} \frac{1}{\sqrt{2\pi\mu_w^2}} \underbrace{e^{-\frac{1}{8\mu_w^2}[e_n(0)-e_{n'}(0)]^2}}_{\to\,\delta_{n,n'} \text{ for } \mu_w\to 0}$$

$$\times\, e^{-\frac{1}{2\mu_w^2}\left[w-e_m(\tau)+\frac{1}{2}(e_n(0)+e_{n'}(0))\right]^2} p_\Lambda(m,n,n')$$

In the limit of accurate measurements $\mu_w = \frac{\mu}{\kappa} \to 0$ the non-diagonal contributions with $n \neq n'$ are suppressed and the remaining Gaussian weights approach delta-functions.

$$p_\Lambda^{\text{pointer}}(w) \to p_\Lambda(w) \quad \text{for } \mu_w \to 0$$

(3) weak measurement

$$p_\Lambda^{\text{pointer}}(w) = \sum_{\substack{m\\ n,n'}} \frac{1}{\sqrt{2\pi\mu_w^2}} \underbrace{e^{-\frac{1}{8\mu_w^2}[e_n(0)-e_{n'}(0)]^2}}_{\to 1 \text{ for } \mu_w \to \infty}$$

$$\times\ e^{-\frac{1}{2\mu_w^2}\left[w - e_m(\tau) + \frac{1}{2}(e_n(0)+e_{n'}(0))\right]^2} p_\Lambda(m, n, n')$$

For a weak measurement, large $\mu_w = \frac{mu}{\kappa}$ the distribution of becomes Gaussian with mean value

$$\langle W\rangle^{\text{weak}} = \mathrm{Tr} H(\lambda(\tau) U_\Lambda \rho(0) U_\Lambda^\dagger - \mathrm{Tr} H(\lambda(0))\rho(0)$$

and variance μ_w^2.
In contrast, in the accurate limit one obtains

$$\langle W\rangle = \sum_n \mathrm{Tr} H(\lambda(\tau) U_\Lambda \Pi_n(0) \rho(0) \Pi_n(0) U_\Lambda^\dagger - \mathrm{Tr} H(\lambda(0))\rho(0)$$

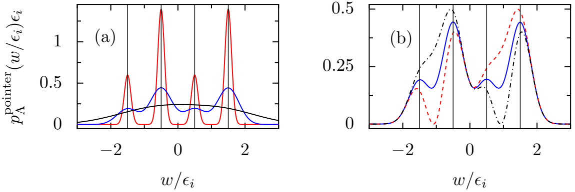
Example

Two-level system undergoing a sudden quench $H_i \to H_f$, with

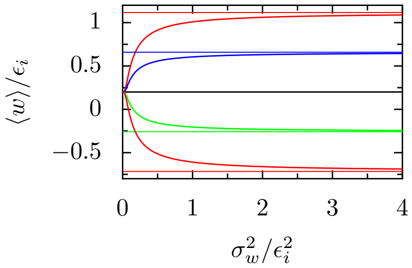
$$H_i = \frac{\epsilon_i}{2}\sigma_z, \quad H_f = \frac{\epsilon_f}{2}\sigma_x$$

initial density matrix (in σ_z-basis):

$$\rho(0) = \begin{pmatrix} p & q \\ q^* & (1-p) \end{pmatrix}, \quad p(1-p) \geq |q|^2$$



$p = 0.7,\ \epsilon_f/\epsilon_i = 2$
(a)
$q = 0$,
$\mu_w^2/\epsilon_i^2 = 0.01, 0.1, 1$
(b)
$q = q_m,\ 0,\ -q_m$
$q_m = \sqrt{p(1-p)} \approx -.458$
$\mu_w^2/\epsilon_i^2 = 0.1$



$p = 0.7$, $\epsilon_f / \epsilon_i = 2$

$q = \pm q_m, \; 0.5 q_m, \; 0, \; -0.5 q_m$

$q_m = \sqrt{p(1-p)} \approx -0.458$

Conclusions

- For projective measurements and "typical" non-degenerate work-values the Crooks relation is equivalent to a generalized detailed balance relation.
- Power-based work measurements do not yield meaningful results for quantum systems.
- For non-degenerate work- and energy-values the Crooks relation is only valid for protocol-independent measurements if they are projective.
- Continuous measurements with Gaussian measurement operators and constant variance obey modified fluctuation relations with protocol-independent modifications.
- The De Chiara-Roncaglia-Paz "work-meter" yields same results as for projective measurements in the accurate limit and the average work as difference of average energies in the weak limit.

Conclusions (cont.)

not discussed

- Fluctuation relations for open quantum systems
- Steady state fluctuation relations

A no-go theorem

Definition

A protocol is TYPICAL if the eigenvalues of the initial and final Hamiltonians $H(\lambda(t)) = \sum_m e_m(t)\Pi_m(t),\ t = 0, \tau$ satisfy the following three conditions:
(A) Any allowed work w corresponds to exactly one pair m, n with $w = e_m(\tau) - e_n(0)$.
(B) The energy-eigenvalues $e_n(0)$ and $e_m(\tau)$ are non-degenerate.
(C) Non-degeneracy of energy distance:
$e_n(t) - e_{n'}(t) = e_k(t) - e_{k'}(t) \ \rightarrow \ n = k,\ n' = k'$.

Theorem

Energy measurement operations $\phi_m^{(t)}$ that are independent of temperature and of the force protocol and, for typical protocols, lead to work statistics satisfying the Crooks relation are always projective: $\phi_m^t(\rho) = \Pi_m(t)\rho\Pi_m(t)$.

A similar statement starting from slightly different condition but leading to the same result was proved in

A no-go theorem (cont.)

B.P. Venkatesh, G. Watanabe, P. Talkner, New J. Phys. **16**, 015032 (2014).

Sketch of a proof of the no-go theorem

With two generalized energy measurements ϕ_n^t the probability density of work becomes:

$$p_\Lambda^\phi(w) = \sum_{m,n} \delta(w - e_m(\tau) + e_n(0)) p_\Lambda^\phi(m,n)$$

$$\begin{aligned} p_\Lambda^\phi(m,n) &= \mathrm{Tr}\phi_m^\tau(U_\Lambda \phi_n^0(\rho(0)) U_\Lambda^\dagger) \\ &= Z^{-1}(0) \sum_{n'} e^{-\beta e_{n'}(0)} \underbrace{\mathrm{Tr}\phi_m^\tau(U_\Lambda \phi_n^0(\Pi_{n'}(0)) U_\Lambda^\dagger)}_{\equiv q_\Lambda(m,n|n') d_{n'}(0)} \end{aligned}$$

Further, we require the validity of the Crooks relation

$$p_\Lambda^\phi(w) = e^{-\beta(\Delta F - w)} p^{\bar{\phi}}_{\bar{\Lambda}(w)}$$

$$p\bar{h}i_n^t(\rho) = \theta^\dagger \phi_n^t(\theta\rho\theta^\dagger)\theta$$

Sketch of a proof of the no-go theorem (cont.)

If an allowed work value uniquely determines a pair of initial and final energies, as it is typically the case, then

$$\text{Crooks relation} \Leftrightarrow Z(0)e^{\beta e_n(0)} p_\Lambda^\phi(m,n) = Z(\tau) e^{\beta e_m(\tau)} p_{\bar{\Lambda}}^{\bar{\phi}}(n,m)$$

$$\sum_{n'} e^{\beta(e_n(0)-e_{n'}(0))} \underbrace{q_\Lambda(m,n|n')d_{n'}(0)}_{\text{independent of } \beta} = \sum_{m'} e^{\beta(e_m(\tau)-e_{m'}(\tau))} \underbrace{q_{\bar{\Lambda}}(n,m|m')d_{m'}(\tau)}_{\text{independent of } \beta}$$

$$\sum_{n'} e^{\beta(e_n(0)-e_{n'}(0))} \underbrace{q_\Lambda(m,n|n')d_{n'}(0)}_{\text{independent of } \beta} = \sum_{m'} e^{\beta(e_m(\tau)-e_{m'}(\tau))} \underbrace{q_{\bar{\Lambda}}(n,m|m')d_{m'}(\tau)}_{\text{independent of } \beta}$$

Typically, also

$$e_n(t) - e_{n'}(t) = e_k(t) - e_{k'}(t) \;\Rightarrow\; n = k,\; n' = k'$$

holds. This implies

$$q_\Xi(m,n|n') = q_\Xi(m|n)\delta_{n,n'} \quad \text{for } \Xi = \Lambda, \bar{\Lambda}$$

hence, ϕ_n^0 and $\bar{\phi}_m^\tau$ error-free operations if they are Λ-independent

From the sums remain the $n' = n$ and $m' = m$ terms yielding the gdb relation

$$q_\Lambda(m|n)d_n(0) = q_{\bar{\Lambda}}(n|m)d_m(\tau)$$

For Λ-independent operations this implies

$$\phi_m^{t*}(\mathbb{1}) = \phi_m^t(\Pi_m(t)) \,, \quad t = 0, \tau \quad (\star)$$

ϕ_m^{t*} is the dual map: $\mathrm{Tr} u \phi_m^t(\rho) = \mathrm{Tr} \phi_m^{t*}(u)\rho$ for all bounded operators u and density matrices ρ.
For the Kraus operators $M_\alpha^m(t)$ defined as
$\phi_m^t(\rho) = \sum_\alpha M_\alpha^m(t)\rho M_\alpha^{m\dagger}(t))$ with $M_\alpha^m(t) = M_\alpha^m(t)\Pi_m(t))$ $(\star)$ implies

$$\sum_\alpha M_\alpha^{m\dagger}(t) M_\alpha^m(t) = \sum_\alpha M_\alpha^m(t) M_\alpha^{m\dagger}(t)$$

yielding for non-degenerate eigenstates ($d_m(t) = \mathrm{Tr}\Pi_m(t) = 1$) that there is only a single $M_\alpha^m(t) = \Pi_m(t)$: measurement must be projective.